

# Infinitely many non-radial sign-changing solutions for a Fractional Laplacian equation with critical nonlinearity

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**Abstract:** In this work, the following fractional Laplacian problem with pure critical nonlinearity is considered

$$\begin{cases} (-\Delta)^s u = |u|^{\frac{4s}{N-2s}} u, & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \end{cases}$$

where  $s \in (0, 1)$ ,  $N$  is a positive integer with  $N \geq 3$ ,  $(-\Delta)^s$  is the fractional Laplacian operator. We will prove that this problem has infinitely many non-radial sign-changing solutions.

## 1 Introduction

We tackle the following Fractional Laplacian problem with pure critical nonlinearity

$$\begin{cases} (-\Delta)^s u = |u|^{p-1} u, & \text{in } \mathbb{R}^N, \\ u \in D^{s,2}(\mathbb{R}^N), \end{cases} \quad (1.1)$$

where  $s \in (0, 1)$ ,  $p = 2^* - 1$ ,  $2^* = \frac{2N}{N-2s}$ ,  $N$  is a positive integer with  $N \geq 3$ , and the space  $D^{s,2}(\mathbb{R}^N)$  is defined as the completion of  $C_0^\infty(\mathbb{R}^N)$  under the norm:

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u} d\xi = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx,$$

where  $\hat{\cdot}$  is Fourier transform.  $(-\Delta)^s$  is a Fractional Laplacian operator which is defined as a pseudo-differential operator:

$$\widehat{(-\Delta)^s u(\xi)} = |\xi|^{2s} \hat{u}(\xi).$$

If  $u$  is smooth enough, it can also be computed by the following singular integral:

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy.$$

Here P.V. is the principal value and  $C_{N,s}$  is a normalization constant. The operator  $(-\Delta)^s$  can be seen as the infinitesimal generators of Lévy stable diffusion processes (see [1]). This operator arises in several areas such as physics, biology, chemistry and finance (see [1, 2]). In recent years, The fractional Laplacians have recently attracted much research interest,

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and there are a lot of results in the literature on the existence of such solutions, e.g., ([3, 5, 6, 7, 8, 9, 10, 14, 19, 20]) and the references therein. In the remarkable work of Caffarelli and Silvestre [7], this nonlocal operator can be defined by the following Dirichlet-to-Neumann map:

$$(-\Delta)^s u(x) = -\frac{1}{k_s} \lim_{y \rightarrow 0^+} \frac{\partial w}{\partial y}(x, y),$$

where,  $k_s = (2^{1-2s}\Gamma(1-s)/\Gamma(s))$  and  $w$  solves the boundary value problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ w(x, 0) = u & \text{on } \mathbb{R}^N. \end{cases}$$

Caffarelli and his coauthors [5, 6] investigated free boundary problems of a fractional Laplacian. The operator was studied by Chang and González [9] in conformal geometry. Silvestre [25] obtained some regularity results for the obstacle problem of the fractional Laplacian. Recently, Fractional Schrödinger equations with respect to wave standing waves were studied in [10, 12, 16, 17, 22, 24]. Very recently, the singularly perturbed problem of fractional Laplacian was considered by Dávila, del Pino and Wei [13], and they recovered various existence results already known for the case  $s = 1$ .

Problem (1.1) arises from looking for a solution for the following fractional Nirenberg problem,

$$P_s(u) = |u|^{\frac{4s}{N-2s}} u, \quad \text{in } \mathbb{S}^N, \quad (1.2)$$

where  $(\mathbb{S}^N, g_{\mathbb{S}^N})$ ,  $n \geq 2$ , be the standard sphere in  $\mathbb{R}^{N+1}$ ,  $P_s$  is an intertwining operator. The reader is referred to [19, 20] for more details on fractional Nirenberg problem. Similar to the case of  $s = 1$ , using the stereo-graphic projection, problem (1.2) can be reduced to problem (1.1).

This idea of this paper is motivated by the recent papers [15, 18], where infinitely many solutions to the Yamabe problem and the Yamabe problem of polyharmonic operator were constructed, respectively.

Our main result in this paper can be stated as follows:

**Theorem 1.1.** *Assume that  $N \geq 3$ , then problem (1.1) has infinitely many non-radial sign-changing solutions.*

We will prove Theorem 1.1 by proving the following result:

**Theorem 1.2.** *Let  $N \geq 3$  and write  $\mathbb{R}^N = \mathbb{C} \times \mathbb{R}^{N-2}$  and  $\xi_j = \sqrt{1-\mu^2}(e^{\frac{2\pi(j-1)}{k}i}, 0, \dots, 0)$ ,  $j = 1, \dots, k$ . Then for any sufficiently large  $k$  there is finite energy solution of the form*

$$u_k(x) = U(x) - \sum_{j=1}^k \mu_k^{-\frac{n-2s}{2}} U(\mu_k^{-1}(x - \xi_j)) + o(1),$$

where

$$\mu_k = \delta_k^{\frac{2}{N-2s}} k^{-3}, U_{x,\Lambda}(y) = c_{N,s} \left( \frac{\Lambda}{1 + \Lambda^2 |y - x|^2} \right)^{\frac{N-2s}{2}}, c_{N,s} > 0, \Lambda > 0, x \in \mathbb{R}^N$$

and  $o(1) \rightarrow 0$  uniformly as  $k \rightarrow \infty$ ,  $\delta_k$  is a positive number which only depends on  $k$ .

**Remark 1.1.** We believe that the similar result should also hold for the following critical problems with the presence of weight:

$$\begin{cases} (-\Delta)^s u = K(x)|u|^{\frac{N+2s}{N-2s}}, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{on } \mathbb{R}^N, \\ u \in D^{s,2}(\mathbb{R}^N). \end{cases}$$

Following the idea in [27], for appropriate weight  $K(x)$ , we can also construct a sequence of non-radial positive solutions for this problem. When  $K(x) = 1$ , it was shown independently by Y. Y. Li and Chen, Li, and Ou [11, 21] that for  $s \in (0, N/2)$  and  $u \in L_{loc}^{\frac{2N}{N-2s}}(\mathbb{R}^N)$  the equation has a unique positive solution  $u(r) > 0$  (see (2.1)) up to scaling and translation.

**Remark 1.2.** In a recent work, W. Long, S.J. Peng and J. Yang [22] obtained infinitely many positive solutions for the following subcritical equation:

$$(-\Delta)^s u + u = k(|x|)u^q, \text{ in } \mathbb{R}^N,$$

where  $q < \frac{N+2s}{N-2s}$ ,  $K(|x|)$  is a positive radial function, and satisfies some asymptotic assumptions at infinity.

We organize this paper as follows. In Section 2, we construct an approximation solution and give the estimates of the error. Section 3 contains a linear result. Section 4 will devote to the detailed calculus and further thoughts on the gluing procedures. The proof the main result will be given in last section.

## 2 Approximation solution and the estimate of the error

We start with the construction of a first approximate solution to problem (1.1). Then we give the precise estimate of the error. It is well known that the radial functions

$$U_{x,\Lambda}(y) = c_{n,s} \left( \frac{\Lambda}{1 + \Lambda^2 |y - x|^2} \right)^{\frac{N-2s}{2}}, \quad c_{n,s} > 0, \Lambda > 0, x \in \mathbb{R}^N, \quad (2.1)$$

are the only solutions to the problem

$$-\Delta u = u^{\frac{N-2s}{N+2s}}, u > 0, \text{ in } \mathbb{R}^N. \quad (2.2)$$

Moreover, the radial solution  $U$  is invariant under the Kelvin type transform

$$\hat{u}(y) = |y|^{2s-N} u \left( \frac{y}{|y|^2} \right). \quad (2.3)$$

That is,  $\hat{U} = U(y)$ . Problem (2.2) is invariant under the Kelvin transform (2.3) (see [4]).

Let

$$w_\mu(y - \xi) = \mu^{-\frac{N-2s}{2}} U \left( \mu^{-1}(y - \xi) \right).$$

Then a simple algebra computation implies that:

**Lemma 2.1.**  $w_\mu(y - \xi)$  is invariant under the Kelvin type transform (2.3) if and only if  $\mu^2 + |\xi|^2 = 1$ .

Let  $k$  be a large positive integer and  $\mu > 0$  be a small concentration parameter such that:

$$\mu = \delta^{\frac{2}{N-2s}} k^{-3} \quad (2.4)$$

where  $\delta$  is a positive parameter that will be fixed later. Let

$$\xi_j = \sqrt{1 - \mu^2} (e^{\frac{2\pi(j-1)}{k}i}, 0, \dots, 0), j = 1, \dots, k.$$

We denote  $U_j(y) := w_\mu(y - \xi_j), j = 1, \dots, k$ , and consider the function

$$U_*(y) := U(y) - \sum_{j=1}^k U_j(y).$$

In order to obtain sign-changing solutions for problem (1.1), we follow the method of [15, 18] and use the number of the bubble solutions  $U_j$  as a parameter. The idea of using the number of bubbles as a parameter was first used by Wei and Yan [27] in constructing infinitely many positive solutions to the prescribing scalar curvature problem. We will prove that when the bubbles number  $k$  is large enough, problem (1.1) has a solution of the form:

$$u(y) = U_*(y) + \phi(y),$$

where  $\phi$  is a small function when compared with  $U$ . If  $u$  satisfies the above form, then problem (1.1) can be rewritten as

$$(-\Delta)^s \phi - p|U_*|^{p-1}\phi + E - N(\phi) = 0, \quad (2.5)$$

where  $p = 2^* - 1, 2^* = \frac{2N}{N-2s}$ , and

$$E = (-\Delta)^s U_* - |U_*|^{p-1} U_*,$$

$$N(\phi) = |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1} U_* - p|U_*|^{p-1} \phi.$$

We will show that for sufficiently large  $k$ , the error term  $E$  will be controlled small enough so that some asymptotic estimate holds. In order to obtain the better control on the error, for a fixed number  $\frac{N}{s} > q > \frac{N}{2s}$ , we consider the following weighted  $L^q$  norm:

$$\|h\|_{**} := \left\| (1+y)^{N+2s-\frac{2N}{q}} h(y) \right\|_{L^q(\mathbb{R}^N)} \quad (2.6)$$

and

$$\|\phi\|_* := \left\| (1+y)^{N-2s} \phi(y) \right\|_{L^\infty(\mathbb{R}^N)}. \quad (2.7)$$

**Lemma 2.2.** *There exist an integer  $k_0$  and a positive constant  $C$  such that for all  $k > k_0$ , the following estimate for  $E$  is true:*

$$\|E\|_{**} \leq C k^{-\frac{N}{q}-2s}. \quad (2.8)$$

**Proof.** We prove this lemma in two steps. Firstly, we consider the error term  $E$  in the exterior region:

$$EX := \bigcap_{j=1}^k B_{\xi_j}^c(\eta/k) := \bigcap_{j=1}^k \{|y - \xi_j| > \eta/k\}.$$

Here  $\eta > 0$  is a positive and small constant, independent of  $k$ . Secondly, we will do it in the interior region:

$$IN = EX^c = \bigcup_{j=1}^k \{|y - \xi_j| \leq \eta/k\},$$

where  $\eta \ll 1$ .

**Step 1.** By the mean value theorem, we have

$$\begin{aligned} E &= (-\Delta)^s U_* - |U_*|^{p-1} U_* \\ &= (-\Delta)^s \left[ U - \sum_{j=1}^k U_j \right] - \left| U - \sum_{j=1}^k U_j \right|^{p-1} \left( U - \sum_{j=1}^k U_j \right) \\ &= U^p - \sum_{j=1}^k U_j^p - \left| U - \sum_{j=1}^k U_j \right|^{p-1} \left( U - \sum_{j=1}^k U_j \right) \\ &= - \left[ p \left| U - t \sum_{j=1}^k U_j \right|^{p-1} \left( - \sum_{j=1}^k U_j \right) + \sum_{j=1}^k U_j^p \right] \\ &= p \left| U - t \sum_{j=1}^k U_j \right|^{p-1} \left( \sum_{j=1}^k U_j \right) - \sum_{j=1}^k U_j^p, \text{ for } t \in (0, 1). \end{aligned} \quad (2.9)$$

Now the exterior region is divided into two parts, that is,

$$A_1 := \{y : |y| \geq 2\} \text{ and } A_2 := \{|y| \leq 2\} \cap \left[ \bigcap_{j=1}^k \{|y - \xi_j| > \eta/k\} \right].$$

For  $y \in A_1$ , one has  $\frac{1}{|y-\xi|} \sim \frac{1}{1+|y|}$ . So,

$$\begin{aligned} |E(y)| &\leq C \left\{ (1 + |y|^2)^{-2s} + \left[ \sum_{j=1}^k \mu^{\frac{N-2s}{2}} (\mu^2 + |y - \xi_j|^2)^{-\frac{N-2s}{2}} \right]^{\frac{4s}{N-2s}} \right\} \\ &\quad \cdot \left[ \sum_{j=1}^k \mu^{\frac{N-2s}{2}} (\mu^2 + |y - \xi_j|^2)^{-\frac{N-2s}{2}} \right] \\ &\leq C \left[ (1 + |y|^2)^{-2s} + \frac{\mu^{2s} k^{\frac{4s}{N-2s}}}{(1 + |y|^2)^{2s}} \right] \cdot \sum_{j=1}^k \frac{\mu^{\frac{N-2s}{2}}}{|y - \xi_j|^{N-2s}} \\ &\leq C \frac{\mu^{\frac{N-2s}{2}}}{(1 + |y|^2)^{2s}} \sum_{j=1}^k \frac{1}{|y - \xi_j|^{N-2s}}. \end{aligned} \quad (2.10)$$

For  $y \in A_2$ , let us consider two cases:

(1) There is a  $i_0 \in \{1, 2, 3, \dots, k\}$  such that  $y$  is closest to  $\xi_{i_0}$ , and far from all the other  $\xi_j$ 's ( $j \neq i_0$ ). Then,

$$|y - \xi_j| \geq \frac{1}{2} |\xi_{i_0} - \xi_j| \sim \frac{|j - i_0|}{k}.$$

(2)  $y$  is far from all  $\xi_i$ 's, that is, there is a  $C_0 > 0$  such that  $|y - \xi_i| \geq C_0 (1 \leq i \leq k)$ .

$$|E(y)| \leq C \left\{ (1 + |y|^2)^{-2s} + \left[ \sum_{j=1}^k \mu^{\frac{N-2s}{2}} (\mu^2 + |y - \xi_j|^2)^{-\frac{N-2s}{2}} \right]^{\frac{4s}{N-2s}} \right\}$$

$$\begin{aligned}
& \cdot \left[ \sum_{j=1}^k \mu^{\frac{N-2s}{2}} (\mu^2 + |y - \xi_j|^2)^{-\frac{N-2s}{2}} \right] \\
& \leq C \left[ (1 + |y|^2)^{-2s} + \left( \frac{\mu^{\frac{N-2s}{2}}}{|y - \xi_{i_0}|^{N-2s}} + \sum_{j \neq i_0} \frac{\mu^{\frac{N-2s}{2}}}{|y - \xi_j|^{N-2s}} \right)^{\frac{4s}{N-2s}} \right] \cdot \sum_{j=1}^k \frac{\mu^{\frac{N-2s}{2}}}{|y - \xi_j|^{N-2s}} \\
& \leq C \left[ (1 + |y|^2)^{-2s} + \left( \mu^{2s} k^{4s} + \max \left\{ \sum_{j \neq i_0} \frac{\mu^{2s} k^{4s}}{|j - i_0|^4}, k^{\frac{4s}{N-2s}} \mu^{2s} \right\} \right) \right] \cdot \sum_{j=1}^k \frac{\mu^{\frac{N-2s}{2}}}{|y - \xi_j|^{N-2s}} \\
& \leq C \frac{\mu^{\frac{N-2s}{2}}}{(1 + |y|^2)^{2s}} \sum_{j=1}^k \frac{1}{|y - \xi_j|^{N-2s}}. \tag{2.11}
\end{aligned}$$

Consequently, from (2.10) and (2.11), in the exterior region, we obtain

$$\begin{aligned}
\|E\|_{**} &= \|(1 + |y|^{(N+2s)q-2N})E^q(y)\|_{L^q(EX)} \\
&\leq C \mu^{\frac{N-2s}{2}} \sum_{j=1}^k \left[ \int_{B_{\xi_j^c}(\eta/k)} \frac{(1 + |y|)^{(N+2s)q-2N}}{(1 + |y|)^{4sq} |y - \xi_j|^{(N-2s)q}} \right]^{1/q} \\
&\leq C \mu^{\frac{N-2s}{2}} k \left[ \int_{\eta/k}^1 \frac{r^{N-1} dr}{r^{(N-2s)q}} + \int_1^{+\infty} r^{-(N+1)} dr \right]^{1/q} \\
&\leq C (k^{1+s-\frac{N}{2}-\frac{N}{q}} + k^{1+3s-\frac{3N}{2}}) \leq C k^{1+s-\frac{N}{2}-\frac{N}{q}}.
\end{aligned}$$

**Step 2.** In the case of the interior region  $IN$ , we easily know that for all  $y \in IN$ , there is  $j \in \{1, 2, 3, \dots, k\}$ , such that  $|y - \xi_j| \leq \eta/k$ . Similar to (2.9), we can obtain

$$E = p \left[ U_j - t \left( - \sum_{j \neq i}^k U_i + U \right) \right]^{p-1} \cdot \left( - \sum_{j \neq i}^k U_i + U \right) + \left( \sum_{j \neq i}^k U_i \right)^p - U^p, \text{ for } t \in (0, 1). \tag{2.12}$$

In order to measure the error term  $E$ , we define

$$\tilde{E}_j = \mu^{\frac{N+2s}{2}} E(\xi_j + \mu y).$$

We observe that

$$\mu^{\frac{N-2s}{2}} U_j(\xi_j + \mu y) = U(y) \text{ and } U_i(y) = \mu^{-\frac{N-2s}{2}} U(\mu^{-1}(y - \xi_i))$$

Thus, for  $i \neq j$ ,

$$\mu^{\frac{N-2s}{2}} U_i(\xi_j + \mu y) = U(y - \mu^{-1}(\xi_i - \xi_j)). \tag{2.13}$$

Note also that  $\mu^{-1}|\xi_i - \xi_j| \sim \frac{|i-j|}{k\mu}$ . Hence, by (2.12) and (2.13), we estimate

$$\begin{aligned}
|\tilde{E}_j(y)| &\leq C \left| U(y) + \sum_{i \neq j} \frac{(k\mu)^{N-2s}}{|j-i|^{N-2s}} + \mu^{\frac{N-2s}{2}} U(\xi_j + \mu y) \right|^{p-1} \\
&\quad \cdot \left( \sum_{i \neq j} \frac{(k\mu)^{N-2s}}{|j-i|^{N-2s}} + \mu^{\frac{N-2s}{2}} U(\xi_j + \mu y) \right) + \sum_{i \neq j} \left( \frac{(k\mu)^{N-2s}}{|j-i|^{N-2s}} \right)^p + \mu^{\frac{N+2s}{2}} U^p(\xi_j + \mu y) \\
&\leq C \left| \mu^{\frac{N-2s}{2}} + \left( \frac{1}{1 + |y|^2} \right)^{\frac{N-2s}{2}} \right|^{p-1} \cdot \mu^{\frac{N-2s}{2}} + \mu^{\frac{N-2s}{2}p} + \mu^{\frac{N+2s}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq C \left| \mu^{\frac{N+2s}{2}} + \frac{\mu^{\frac{N-2s}{2}}}{1 + |y|^{4s}} \right| \\
&\leq C \frac{\mu^{\frac{N-2s}{2}}}{1 + |y|^{4s}}.
\end{aligned}$$

Therefore, we can get

$$\begin{aligned}
\|E\|_{**(|x-\xi_j|<\eta/k)} &= \left( \int_{|x-\xi_j|<\eta/k} (1 + |x|)^{(N+2sq)-2N} \mu^{-\frac{N+2s}{2}q} \tilde{E}^q \left( \frac{x - \xi_j}{\mu} \right) dx \right)^{\frac{1}{q}} \\
&\leq C \left[ \int_{|y|\leq\eta/(k\mu)} \left| \mu^{\frac{N}{q}-\frac{N+2s}{2}} \tilde{E}_j(y) \right|^q dy \right]^{1/q} \\
&\leq C \left[ \mu^{N-2qs} \int_0^{\eta/(k\mu)} \frac{r^{N-1}}{1 + r^{4qs}} dr \right]^{1/q} \\
&\leq C \mu^{2s} k^{-\frac{N}{q}+4s} \leq C k^{-\frac{N}{q}-2s}.
\end{aligned}$$

Finally, by combining the estimates in the exterior region and interior region together, we have

$$\begin{aligned}
\|E\|_{**} &\leq \|E\|_{**(EX)} + \|E\|_{**(IN)} \\
&\leq \|E\|_{**(EX)} + \sum_{j=1}^k \|E\|_{**(IN)(|x-\xi_j|<\eta/k)} \\
&\leq C k^{-\frac{N}{q}-2s}.
\end{aligned}$$

□

### 3 A linear result

We consider the operator  $L_0$  defined as

$$L_0(\phi) := (-\Delta)^s \phi - p U^{p-1} \phi, \text{ with } p = m^* - 1.$$

We can know that (see [14]) the solution space for the homogeneous equation  $L_0(\phi) = 0$  is spanned by the  $n + 1$  functions,

$$v_i = \partial_{y_i} U, \quad i = 1, 2, \dots, N; \quad v_{N+1} = x \cdot \nabla U + \frac{n-2s}{2} U.$$

This section is devoted to establishing an invertibility theory for

$$L_0(\phi) = h \text{ in } \mathbb{R}^N. \quad (3.1)$$

It is worth noting that the  $L^p$  to  $W^{2s,p}$  estimate does not hold for all  $p$  in this fractional framework (see Remarks 7.1 and 7.2 in [23]). But we have the following Lemma 3.1. This is an important ingredient in the proof of Lemma 3.2.

**Lemma 3.1** ([23, 26]). *Let  $\Omega \subset \mathbb{R}^N$  be a bounded  $C^{1,1}$  domains,  $s \in (0, 1)$ ,  $n > 2s$ ,  $g \in C(\bar{\Omega})$ , and  $u$  be the solution of*

$$\begin{cases} (-\Delta)^s u = g, & \text{in } \Omega, \\ u > 0, & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.2)$$

- (i) For each  $1 \leq r < \frac{n}{n-2s}$ , there exists a constants  $C$  depending only  $n, s, r$  and  $|\Omega|$ , such that

$$\|u\|_{L^r(\Omega)} \leq C\|g\|_{L^1(\Omega)}, \quad r < \frac{n}{n-2s}.$$

- (ii) Let  $1 \leq p < \frac{n}{2s}$ . Then there exists a constant  $C$  depending only on  $n, s$  and  $p$  such that

$$\|u\|_{L^q(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \quad \text{where } q = \frac{np}{n-2ps}.$$

- (iii) Let  $\frac{n}{2s} \leq p < \infty$ . Then, there exists a constants  $C$  depending only on  $n, s, p$  and  $|\Omega|$  such that

$$\|u\|_{C^\beta(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \quad \text{where } \beta = \min \left\{ s, 2s - \frac{n}{p} \right\}.$$

**Lemma 3.2.** Let  $h(y)$  be a function such that  $\|h\|_{**} < \infty$ . Assume that  $\frac{N}{2s} < q < \frac{N}{s}$ , and

$$\int_{\mathbb{R}^N} v_l h = 0, \forall l = 1, 2, \dots, N+1.$$

Then problem (3.1) has a unique solution satisfying  $\|\phi\|_* < \infty$  and

$$\int_{\mathbb{R}^N} U^{p-1} v_l h = 0, \forall l = 1, 2, \dots, N+1.$$

Furthermore, there exists a constant  $C$  which depends on  $q$  and  $N$  such that

$$\|\phi\|_* \leq C\|h\|_{**}.$$

**Proof.** Set

$$H = \left\{ \phi \in \mathcal{D}^{s,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} U^{p-1} v_l \phi dx = 0, \forall l = 1, 2, \dots, N+1 \right\}.$$

Then  $H$  is a Hilbert space equipped with the following norm:

$$\|u\|_H := \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(y)|^2 dy.$$

Furthermore, for all  $\psi \in H$ , it is easy to show that

$$(L_0 \phi, \psi)_H = (\phi, L_0 \psi)_H.$$

Let  $r = \frac{2N}{N+2s}, r' = \frac{2N}{N-2s} = p+1$ . Since  $\|h\|_* \leq \infty$ , the Hölder inequality implies

$$\begin{aligned} \|h\|_r &\leq \left[ \int_{\mathbb{R}^N} \left| h^r (1+|y|)^{(N+2s)r - \frac{2Nr}{q}} \cdot (1+|y|)^{-(N+2s)r + \frac{2Nr}{q}} \right| dy \right]^{\frac{1}{r}} \\ &\leq \left[ \int_{\mathbb{R}^N} |h|^q (1+|y|)^{(N+2s)q - 2N} dy \right]^{1/q} \left[ \int_{\mathbb{R}^N} (1+|y|)^{-2N} \right]^{\frac{1}{r} - \frac{1}{q}} \\ &\leq C\|h\|_{**} < \infty, \end{aligned} \tag{3.3}$$

and

$$\|U^{p-1} \phi\|_r \leq \left( \int_{\mathbb{R}^N} |\phi|^{r \cdot \frac{N+2s}{N-2s}} \right)^{\frac{N-2s}{(N+2s)r}} \cdot \left( \int_{\mathbb{R}^N} |U|^{(p-1)r \cdot \frac{N+2s}{4s}} \right)^{\frac{4s}{(N+2s)r}}$$



$$\begin{aligned}
&= \|\phi\|_{p+1} \cdot \left( \int_{\mathbb{R}^N} |U|^{\frac{2N}{N-2s}} \right)^{\frac{2s}{N}} \\
&\leq C \|\phi\|_{p+1} \leq C \|\phi\|_H < \infty \text{ (by fractional Sobolev inequality)}. \tag{3.4}
\end{aligned}$$

For  $f \in L^r$ , using (3.3) and (3.4), the weak solution can be considered by the following equation

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \phi \cdot (-\Delta)^{\frac{s}{2}} \psi + \int_{\mathbb{R}^N} f \psi = 0, \text{ for all } \psi \in H. \tag{3.5}$$

Define the functional  $A_f : H \rightarrow \mathbb{R}$  as follow:

$$A_f(\psi) = \int_{\mathbb{R}^N} f \psi,$$

and we easily know that

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \phi \cdot (-\Delta)^{\frac{s}{2}} \psi = A_f(\psi). \tag{3.6}$$

Furthermore, using Hölder inequality again, one has

$$|A_f(\psi)| \leq \|f\|_r \|\psi\|_{p+1} \leq C \|f\|_r \|\psi\|_H,$$

so this shows that  $A_f$  is a bounded linear functional on the Hilbert space  $H$ . Applying the Riesz representation theorem, there is a unique  $\phi \in H$  such that

$$A_f(\psi) = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \phi \cdot (-\Delta)^{\frac{s}{2}} \psi.$$

Hence, according to the functional  $A_f$ , one can define a new operator  $A : L^r \rightarrow H$  by

$$A(f) = \phi \text{ and } \langle A(f), \psi \rangle_H = \langle f, \psi \rangle, \forall \psi \in H.$$

Then (3.1) can be formulated as

$$\phi = A(h) + A(pU^{p-1}\phi) = A(h) + A(\tau(\phi)),$$

where  $\tau : H \rightarrow L^r, \phi \rightarrow pU^{p-1}\phi$ , is a compact mapping, thanks to local compactness of Sobolev's embedding and the fact that  $U^{p-1} = O(|y|^{-4})$ .

Let  $B = A \circ \tau$ , then we easily show that  $B$  is the operator from  $H$  to  $H$ , and also compact, self-adjoint. Hence problem (3.1) can be equivalent to

$$(I - B)\phi = A(h).$$

Now Fredholm alternative theorem tell us that the above equation has a solution if and only if

$$\forall v \in \ker(I - B), \quad (I - B)v = 0 = A(0).$$

Therefore, we get  $h \equiv 0$  and

$$A(0) \in R(I - B) = (\ker(I - B^*))^\perp = (\ker(I - B))^\perp.$$

So problem (3.1) be simplified as the homogeneous formula, namely,

$$L_0(v) = 0,$$

where  $v$  can be denoted by the sum of  $v_i$ 's, that is,

$$v(y) = \sum_{j=1}^{N+1} a_j v_j(y),$$

with constants  $a_1, a_2, \dots, a_{N+1}$ .

By the definition of  $H$ , we have

$$0 = \int_{\mathbb{R}^N} U^{p-1} v_j \cdot v = a_j \int_{\mathbb{R}^N} U^{p-1} v_j,$$

which yields the vanishing terms

$$a_j = 0, j = 1, 2, \dots, N+1, \text{ and } v \equiv 0, \ker(I - B) = \{0\}.$$

Consequently, the orthogonal terms  $R(I - B) = H$ ; this implies the existence of  $\phi$  by

$$(I - B)\phi = A(h),$$

and the uniqueness of  $\phi$  by

$$\ker(I - B) = \{0\}.$$

In the following, we will prove that

$$\|\phi\|_* \leq C\|h\|_{**}.$$

By Lemma 3.1(iii), we have

$$\|\phi\|_{L^\infty(B)} \leq C\|(1 + |y|^{N+2s-\frac{2N}{q}})h\|_{L^q(\mathbb{R}^N)}. \quad (3.7)$$

Now let us consider Kelvin's transform of  $\phi$ ,

$$\tilde{\phi}(y) = |y|^{2s-N}\phi(|y|^{-2}y).$$

Then we easily see that  $\tilde{\phi}$  satisfies the equation

$$(-\Delta)^s \tilde{\phi} - pU^{p-1}\tilde{\phi} = \tilde{h}, \text{ in } \mathbb{R}^N, \quad (3.8)$$

where  $\tilde{h}(y) = |y|^{-2s-N}h(|y|^{-2}y)$ . Note that

$$\|\tilde{h}\|_{L^q(|y|<2)} = \||y|^{N+2s-\frac{2N}{q}}h\|_{L^q(|y|>1/2)} \leq C\|(1 + |y|)^{N+2s-\frac{2N}{q}}h\|_{L^q(\mathbb{R}^N)} \quad (3.9)$$

Applying Lemma 3.1 to (3.8), and by (3.9) we obtain

$$\|\tilde{\phi}\|_{L^\infty(B)} \leq C\|\tilde{h}\|_{L^q(B_2)} \leq C\|(1 + |y|)^{N+2s-\frac{2N}{q}}h\|_{L^q(\mathbb{R}^N)}. \quad (3.10)$$

But

$$\|\tilde{\phi}\|_{L^\infty(B)} = \||y|^{n-2s}\phi\|_{L^\infty(\mathbb{R}^N \setminus B_1)}. \quad (3.11)$$

Finally, (3.7), (3.10) and (3.11) imply that

$$\|\phi\|_* \leq C\|h\|_{**}.$$

□

## 4 A gluing procedure

Let  $\zeta(s)$  be a smooth function satisfying

$$\zeta = \begin{cases} 1 & 0 \leq s < \frac{1}{2}; \\ \text{smooth} & \frac{1}{2} \leq s \leq 1; \\ 0 & s > 1. \end{cases}$$

The cut-off functions are defined by

$$\zeta_j = \begin{cases} \zeta(k\eta^{-1}|y|^{-2}|(y - |y|^2\xi_j)|) & \text{if } |y| \geq 1; \\ \zeta(k\eta^{-1}|(y - \xi_j)|) & \text{if } |y| < 1. \end{cases}$$

Then a simple algebra computation implies that

$$\zeta_j(y) = \zeta_j(|y|^{-2}y), \quad \text{supp}(\zeta_j) \subset \{y : |y - \xi_j| \leq \eta/k\}, j = 1, 2, \dots, k.$$

Let  $\phi = \sum_{k=1}^k \tilde{\phi}_j + \psi$ ,  $\bar{y} = (y_1, y_2)$ ,  $y' = (y_3, \dots, y_N)$ , and we assume

$$\tilde{\phi}_j(\bar{y}, y') = \tilde{\phi}_1(e^{-\frac{2\pi(j-1)}{k}}\sqrt{-1}\bar{y}, y'), \quad j = 1, \dots, k, \quad (4.1)$$

$$\tilde{\phi}_1(y) = |y|^{2s-N} \tilde{\phi}_1(|y|^{-2}y), \quad (4.2)$$

$$\tilde{\phi}_1(y_1, \dots, y_s, \dots, y_N) = \tilde{\phi}_1(y_1, \dots, -y_s, \dots, y_N), s = 2, 3, \dots, N, \quad (4.3)$$

and

$$\|\phi_1\|_* \leq \rho \text{ with } \rho \ll 1, \quad (4.4)$$

where  $\phi_1(y) := \mu^{\frac{N-2s}{2}} \tilde{\phi}_1(\xi_1 + \mu y)$ . Now, due to the cut-off function, problem (2.5) can be split into the following system:

$$\begin{cases} (-\Delta)^s \tilde{\phi}_j - p|U_*|^{p-1} \zeta_j \tilde{\phi}_j + \zeta_j \left[ -p|U_*|^{p-1} \psi + E - N \left( \tilde{\phi}_j + \sum_{i \neq j} \tilde{\phi}_i + \psi \right) \right] = 0, & j = 1, \dots, k, \\ (-\Delta)^s \psi - pU^{p-1} \psi + \left[ -p(|U_*|^{p-1} - U^{p-1}) \left( 1 - \sum_{j=1}^k \zeta_j \right) + pU^{p-1} \left( \sum_{j=1}^k \zeta_j \right) \right] \psi \\ - p|U_*|^{p-1} \sum_{j=1}^k (1 - \zeta_j) \tilde{\phi}_j + \left( 1 - \sum_{j=1}^k \zeta_j \right) \left( E - N \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right) = 0. \end{cases} \quad (4.5)$$

In order to obtain the existence and uniqueness of solution  $\psi$  for the equation in (4.5), we can simplify the last equation in (4.5) to

$$(-\Delta)^s \psi - pU^{p-1} \psi + (V_1 + V_2) \cdot \psi + M(\psi) = 0, \quad (4.6)$$

where

$$V_1 = -p(|U_*|^{p-1} - U^{p-1}) \left( 1 - \sum_{j=1}^k \zeta_j \right), \quad V_2 = pU^{p-1} \left( \sum_{j=1}^k \zeta_j \right), \quad (4.7)$$

$$M(\psi) = -p|U_*|^{p-1} \sum_{j=1}^k (1 - \zeta_j) \tilde{\phi}_j + \left( 1 - \sum_{j=1}^k \zeta_j \right) \left( E - N \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right), \quad (4.8)$$

and

$$N(\phi) = |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\phi. \quad (4.9)$$

**Lemma 4.1.** Assume that  $\tilde{\phi}_j$  satisfy the conditions (4.1)-(4.4). Then there exist constants  $k_0, \rho_0, C$  such that for all  $k \geq k_0$  and  $\rho < \rho_0$  problem (4.6) has a unique solution  $\psi = \Psi(\phi_1)$  which satisfies the following symmetrical properties

$$\psi_1(\bar{y}, y_3, \dots, y_l, \dots, y_N) = \psi(\bar{y}, y_3, \dots, -y_l, \dots, y_N); \quad (4.10)$$

$$\psi(y) = |y|^{2s-N} \psi(|y|^{-2}y); \quad (4.11)$$

$$\psi(\bar{y}, y') = \psi(e^{\frac{2\pi j}{k}\sqrt{-1}}\bar{y}, y'), \quad j = 1, \dots, k-1. \quad (4.12)$$

Furthermore

$$\|\psi\|_* \leq C \left[ k^{1+s-\frac{N}{2}-\frac{N}{q}} + \|\phi_1\|_*^2 \right]$$

and the operator  $\Psi$  satisfies the Lipschitz property

$$\|\Psi(\phi_1^1) - \Psi(\phi_1^2)\|_* \leq C \|\phi_1^1 - \phi_1^2\|_*.$$

**Proof.** Firstly, consider linear problem (3.1), and assume that  $h$  satisfies the properties (4.10)-(4.12), namely,

$$h_1(\bar{y}, y_3, \dots, y_l, \dots, y_N) = h(\bar{y}, y_3, \dots, -y_l, \dots, y_N); \quad (4.13)$$

$$h(y) = |y|^{2s-N} h(|y|^{-2}y); \quad (4.14)$$

$$h(\bar{y}, y') = h(e^{\frac{2\pi j}{k}\sqrt{-1}}\bar{y}, y'), \quad j = 1, \dots, k-1. \quad (4.15)$$

We will prove that (3.1) has a unique bounded solution  $\psi = T(h)$  and there exists a constant  $C$  depending on  $q$  and  $N$  such that

$$\|\psi\|_* \leq C \|h\|_{**}$$

On the basis of the results in Lemma 3.2, we need to verify that

$$(h, v_l) = \int_{\mathbb{R}^N} h v_l = 0, \quad \forall l = 1, 2, \dots, N+1.$$

By the assumption (4.13) that  $h$  is an even function, and the oddness of  $v_l = \frac{\partial U}{\partial y_l}$ , we easily know that  $(h, v_l) = 0$  for  $l = 3, \dots, N$ .

For the cases  $l = 1, 2$ , we consider the vector integral

$$I = \int_{\mathbb{R}^N} h \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} dy = c_N \int_{\mathbb{R}^N} \frac{h(y)}{(1 + |y|^2)^{\frac{N}{2}-1+s}} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} dy.$$

Set

$$(\tilde{z}, z') = (e^{\frac{2\pi j}{k}\sqrt{-1}}\bar{y}, y').$$

From the assumption (4.14), it is easy to see that

$$e^{\frac{2\pi j}{k}\sqrt{-1}} I = c_N \int_{\mathbb{R}^N} \frac{h(y)}{(1 + |y|^2)^{\frac{N}{2}-1+s}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} e^{\frac{2\pi j}{k}\sqrt{-1}} dy \quad (4.16)$$

$$= c_N \int_{\mathbb{R}^N} \frac{h(z)}{(1 + |z|^2)^{\frac{N}{2}-1+s}} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} dz \quad (4.17)$$

$$= I, \quad (4.18)$$

which implies  $I = 0$ , since  $e^{\frac{2\pi j}{k}\sqrt{-1}} \neq 0$  for  $k \geq 2$ .

For the case  $l = N + 1$ , let us consider the following function  $I(\lambda)$ ,  $\lambda > 0$ ,

$$I(\lambda) = \lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} U(\lambda y) h(y) dy.$$

Using the transformation  $z = |y|^{-2}y$ , we obtain

$$\begin{aligned} I(\lambda) &= \lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} U(\lambda y) h(y) dy \\ &= \lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} U(\lambda |y|^{-2}y) h(|y|^{-2}y) d(|y|^{-2}y) \\ &= \lambda^{-\frac{N-2s}{2}} \int_{\mathbb{R}^N} U(\lambda^{-1}y) h(y) dy \\ &= I(\lambda^{-1}) := g(\lambda). \end{aligned} \quad (4.19)$$

This implies

$$I'(1) = g'(1) = -\frac{1}{\lambda^2} I' \left( \frac{1}{\lambda} \right) \Big|_{\lambda=1} = -I'(1).$$

So

$$0 = I'(1) = (h, v_{N+1}).$$

Therefore, by Lemma 3.2, we have

$$\|T(h)\|_* = \|\psi\|_* \leq C \|h\|_{**}.$$

Next we will show that problem (4.6) has a unique solution. Taking  $h = (V_1 + V_2)\psi + M(\psi)$ , then we write our problem in fixed point form as

$$\mathcal{M}(\psi) := -T((V_1 + V_2) + M(\psi)) = \psi, \quad \psi \in X,$$

where  $X$  denotes the linear space with bounded norm  $\|\cdot\|_*$  and satisfies all symmetry properties in Lemma 4.1. Now noting that

$$[(V_1 + V_2 \cdot \psi) + M(\psi)](y) = |y|^{-(N+2s)} [(V_1 + V_2 \cdot \psi) + M(\psi)] (|y|^{-2}y),$$

we can show that

$$\psi_l(y) = \psi(\bar{y}, y_3, \dots, -y_l, \dots, y_N), l = 3, 4, \dots, N; \quad (4.20)$$

$$\psi_{N+1}(y) = |y|^{2s-N} h(|y|^{-2}y); \quad (4.21)$$

and

$$\psi^j(y) = \psi(e^{\frac{2\pi j}{k}\sqrt{-1}} \bar{y}, y'), \quad j = 1, \dots, k-1 \quad (4.22)$$

satisfy problem (3.1). Hence, by the unique result of Lemma 3.1, we have

$$\psi = \psi_l = \psi^j = \psi_{N+1},$$

which are exactly the symmetries required in Lemma 4.1.

The rest of this section is devoted to proving that  $\mathcal{M}$  is a contraction mapping. To do this, we must make a series of estimates of  $V_1, V_2, M$ , respectively.

Recall that

$$V_1 = -p(|U_*|^{p-1} - U^{p-1}) \left( 1 - \sum_{j=1}^k \zeta_j \right);$$

and the assumptions of the cut-off functions  $\zeta_j$  imply that  $\text{supp} V_1 \subset EX$

Now by taking the similar estimate as in the discussion of Step 1 in Lemma 2.2, for all  $y \in EX$ , there is a  $s \in (0, 1)$  such that

$$\begin{aligned} |V_1(y)\psi(y)| &= \left| V_1(y)\psi(y)(1 + |y|^{N-2s}) \frac{1}{1 + |y|^{N-2s}} \right| \\ &\leq C|V_1(y)U(y)| \cdot |(1 + |y|^{N-2s})\psi| \\ &\leq C|V_1(y)U(y)| \|\psi(y)\|_* \\ &\leq C|U^{p-1}(y) - U_*^{p-1}(y)| \cdot \|\psi\|_* U(y) \\ &\leq \left| U(y) - s \sum_{j=1}^k U_j(y) \right|^{p-2} \left[ \sum_{j=1}^k U_j(y) \right] \cdot \|\psi\|_* U(y) \end{aligned} \tag{4.23}$$

Since  $EX = A_1 \cup A_2$ , for  $y \in A_1$ , we get  $\mu^2 + |y - \xi_j|^2 \sim 1 + |y|^2$ . So

$$\sum_{j=1}^k U_j(y) \leq Ck\mu^{\frac{N-2s}{2}}U \leq Ck^{3s+1-\frac{3N}{2}}U(y) \leq CU(y). \tag{4.24}$$

For  $y \in A_2$ , we obtain

$$\begin{aligned} \sum_{j=1}^k U_j(y) &= \sum_{j=1}^k \mu^{\frac{N-2s}{2}} \left( \frac{1}{\mu^2 + |y - \xi_j|^2} \right)^{\frac{N-2s}{2}} \\ &\leq Ck \cdot k^{\frac{N-2s}{2}} \mu^{\frac{N-2s}{2}} \leq Ck^{1+2s-N} \leq CU(2) \leq CU(y). \end{aligned} \tag{4.25}$$

From (4.24) and (4.25), we infer

$$\sum_{j=1}^k U_j(y) \leq CU(y), \text{ for all } y \in EX. \tag{4.26}$$

Using (4.26), we can amplify (4.23), and obtain

$$\begin{aligned} |V_1(y)\psi(y)| &\leq C\|\psi\|_* U^{p-1} \left( \sum_{j=1}^k U_j(y) \right) \\ &\leq \|\psi\|_* \left( \frac{1}{(1 + |y|^2)} \right)^{2s} \sum_{j=1}^k \frac{\mu^{\frac{N-2s}{2}}}{|y - \xi_j|^{N-2s}} \end{aligned} \tag{4.27}$$

By taking the similar estimate as in Step 1 in Lemma 2.2, we see that

$$\|V_1\psi\|_{**} \leq Ck^{1+s-\frac{N}{2}-\frac{N}{q}}.$$

Now we turn to the estimate  $V_2\psi$ . Recall that

$$V_2 = pU^{p-1} \left( \sum_{j=1}^k \zeta_j \right).$$

The assumptions of the cut-off functions  $\zeta_j$  imply that  $\text{supp} V_2$  lies in the annular, that is,

$$\text{supp} V_2 \subset IN \subset \left\{ y : ||y| - \sqrt{1 - \mu^2}| \leq \frac{\eta}{k} \right\} := AN.$$

Therefore

$$\begin{aligned} \|V_2\psi\|_{**} &\leq C\|\psi\|_* \sum_{j=1}^k \|U^p \zeta_j\|_{**(|y-\xi_j| \leq \eta/k)} \\ &\leq \|\psi\|_* \cdot m(AN) \\ &\leq C\|\psi\|_* \cdot k^{1-N}, \end{aligned} \tag{4.28}$$

where  $m(\cdot)$  denotes Lebesgue measure.

We know that  $M$  is a nonlinear operator, so the estimate of  $M$  will be more complicated. For convenience sake, we introduce the following notations.

Set

$$\begin{aligned} M_1 &= -p|U_*|^{p-1} \sum_{j=1}^k (1 - \zeta_j) \tilde{\phi}_j, \\ M_2 &= \left( 1 - \sum_{j=1}^k \zeta_j \right) E, \end{aligned}$$

and

$$M_3(\psi) = - \left( 1 - \sum_{j=1}^k \zeta_j \right) N \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right).$$

Then the nonlinear operator can be denoted by

$$M(\psi) = M_1 + M_2 + M_3(\psi).$$

For  $M_1$ , applying the estimate of the exterior region and (4.1), (4.2) and (4.4), we get

$$\|M_1\|_* \leq C \sum_{j=1}^k \| |U_*|^{p-1} \tilde{\phi}_j \|_{**(|y-\xi_j| > \eta/k)} \leq C k^{1+s-\frac{N}{2}-\frac{N}{q}}. \tag{4.29}$$

The discussion for  $M_2$  is the same as that for the error term  $E$ .

For  $M_3(\psi)$ , we easily see that  $\text{supp} M_3(\psi) \subset EX$ . By the definition of  $N$  and the mean value theorem, there exist  $s, t \in (0, 1)$  such that

$$\begin{aligned} &\left| N \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right| \\ &= \left| \left| U_* + \sum_{j=1}^k \tilde{\phi}_j + \psi \right|^{p-1} \cdot \left( U_* + \sum_{j=1}^k \tilde{\phi}_j + \psi \right) - |U_*|^{p-1} U_* - p|U_*|^{p-1} \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right| \end{aligned}$$

$$\begin{aligned}
&= p \left| \left| U_* + s \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right|^{p-1} \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) - |U_*|^{p-1} \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right| \\
&= sp \left| \left| U_* + st \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right|^{p-2} \left| \sum_{j=1}^k \tilde{\phi}_j + \psi \right|^2 \right|.
\end{aligned}$$

In the exterior region, using the discussion of Step 1 in Lemma 2.2, we infer

$$\left| \sum_{j=1}^k \tilde{\phi}_j(y) \right| \leq C \|\phi_1\|_* U(y) \cdot \left( \sum_{j=1}^k \frac{\mu^{\frac{N-2}{2}}}{|y - \xi_j|^{N-2s}} \right).$$

So

$$\begin{aligned}
\|M_3(\psi)\|_{**} &= \left\| \left( 1 - \sum_{j=1}^k \zeta_j \right) N \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right\|_{**} \\
&\leq C \left\| N \left( \sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right\|_{**(EX)} \\
&\leq C \left\| \left[ |U_*| + \left| \sum_{j=1}^k \tilde{\phi}_j \right| + |\psi| \right]^{p-2} \left[ \sum_{j=1}^k \tilde{\phi}_j^2 + |\psi|^2 \right] \right\|_{**(EX)} \\
&\leq C \|\phi_1\|_*^2 \left\| U^p \cdot \sum_{j=1}^k \frac{\mu^{\frac{N-2}{2}}}{|y - \xi_j|^{N-2s}} \right\|_{**(EX)} + C \|\psi\|_*^2 \|U^p\|_{**(EX)} \\
&\leq C \|\phi_1\|_*^2 k^{1+s-\frac{N}{2}-\frac{N}{q}} + C \|\psi\|_*^2.
\end{aligned}$$

Combining the above estimates with respect to  $M_1, M_2$  and  $M_3$ , one has

$$\begin{aligned}
\|M(\psi)\|_{**} &\leq \|M_1\|_{**} + \|M_2\|_{**} + \|M_3\|_{**} \\
&\leq C k^{1+s-\frac{N}{2}-\frac{N}{q}} + C \|\phi_1\|_*^2 k^{1+s-\frac{N}{2}-\frac{N}{q}} + C \|\psi\|_*^2.
\end{aligned}$$

Next we will use Banach fixed point theorem to prove Lemma 4.1. So we must show that  $\mathcal{M}$  is a contraction mapping from the small ball in  $X$  to the ball itself.

By the mean value theorem, there exist some  $s, t \in (0, 1)$  such that

$$\begin{aligned}
\|M(\psi_1) - M(\psi_2)\|_{**} &= \|M_3(\psi_1) - M_3(\psi_2)\|_{**} \\
&\leq C \left\| N \left( \sum_{j=1}^k \tilde{\phi}_j + \psi_1 \right) - N \left( \sum_{j=1}^k \tilde{\phi}_j + \psi_2 \right) \right\|_{**(EX)} \\
&\leq \left\| p \left| U_* + \sum_{j=1}^k \tilde{\phi}_j + s(\psi_1 - \psi_2) \right|^{p-1} (\psi_1 - \psi_2) - p |U_*|^{p-1} (\psi_1 - \psi_2) \right\|_{**(EX)} \\
&\leq \left\| \left| U_* + t \sum_{j=1}^k \tilde{\phi}_j + st(\psi_1 - \psi_2) \right|^{p-2} \left| \sum_{j=1}^k \tilde{\phi}_j + s(\psi_1 - \psi_2) \right| \cdot |\psi_1 - \psi_2| \right\|_{**(EX)} \\
&\leq C (\|\phi_1\|_* + \|\psi_1 - \psi_2\|_*) \|\psi_1 - \psi_2\|_* \cdot \|U^p\|_{**(EX)}
\end{aligned}$$



$$\leq C\|\psi_1 - \psi_2\|_{*(EX)}.$$

Then, we have

$$\begin{aligned}\|\mathcal{M}(\psi_1 - \psi_2)\| &= \|-T[(V_1 + V_2)(\psi_1 - \psi_2) + (M(\psi_1 - \psi_2))]\|_* \\ &\leq C\|((V_1 + V_2)(\psi_1 - \psi_2))\|_{**} + C\|(M(\psi_1) - M(\psi_2))\|_{**} \\ &\leq C(k^{1-\frac{N}{q}} + \rho)\|\psi_1 - \psi_2\|_*.\end{aligned}$$

Let  $k_0$  be a large positive integer and  $\rho_0$  be small enough such that for each  $k > k_0$  and  $\rho < \rho_0$

$$C(k^{1+s-\frac{N}{2}-\frac{N}{q}} + \rho) < 1.$$

Finally, we use Banach fixed point theorem to complete proof.  $\square$

In the following, we will study the first series in (4.5):

$$(-\Delta)^s \tilde{\phi}_j - p|U_*|^{p-1} \zeta_j \tilde{\phi}_j + \zeta_j \left[ -p|U_*|^{p-1} \psi + E - N \left( \tilde{\phi}_j + \sum_{i \neq j} \tilde{\phi}_i + \psi \right) \right] = 0, \quad j = 1, \dots, k.$$

In fact, by the assumptions (4.1) and (4.2), we can make the changing of variables to simplify the above equations as a single equation:

$$(-\Delta)^s \tilde{\phi}_1 - p|U_*|^{p-1} \zeta_1 \tilde{\phi}_1 + \zeta_1 \left[ -p|U_*|^{p-1} \psi + E - N \left( \tilde{\phi}_1 + \sum_{i \neq 1} \tilde{\phi}_i + \psi \right) \right] = 0, \quad j = 1, \dots, k. \quad (4.30)$$

For convenience sake, we introduce the following notations:

$$\begin{aligned}\mathcal{N}(\phi_1) &:= p(|U_1|^{p-1} - |U_*|^{p-1} \zeta_1) \tilde{\phi}_1 + \zeta_1 \left[ -p|U_*|^{p-1} \psi + E - N \left( \tilde{\phi}_1 + \sum_{i \neq 1} \tilde{\phi}_i + \psi \right) \right], \\ \tilde{h} &:= \zeta_1 E + \mathcal{N}(\phi_1).\end{aligned}$$

We easily see that  $\tilde{h}$  satisfy the following properties:

$$\tilde{h}(y_1, y_2, \dots, y_l, \dots, y_N) = \tilde{h}(y_1, y_2, \dots, -y_l, \dots, y_N), \quad l = 2, 3, \dots, N; \quad (4.31)$$

$$\tilde{h}(y) = |y|^{-2s-N} \tilde{h}(|y|^{-2} y). \quad (4.32)$$

Then Eq. (4.30) can be reduced to

$$[(-\Delta)^s - p|U_1|^{p-1} \zeta_1] \tilde{\phi}_1 + \tilde{h} = 0 \quad (4.33)$$

According to the definition of  $\mu$ , we see that  $\mu$  is related to  $\delta$ . Hence

$$c_{N+1}(\delta) := \frac{\int_{\mathbb{R}^N} (\zeta_1 E + \mathcal{N}(\phi_1)) \tilde{v}_{N+1}}{\int_{\mathbb{R}^N} U_1^{p-1} \tilde{v}_{N+1}^2}$$

is also related to  $\delta$ . Using translating and scaling, we easily know that Eq. (4.33) is equivalent to (3.1). In order to obtain the unique existence of (4.33), by the results in Lemma 3.2, we need to verify that

$$\int_{\mathbb{R}^N} \tilde{h} \tilde{v}_l = \int_{\mathbb{R}^N} h v_l = 0, \quad \forall l = 1, 2, \dots, N+1.$$

Moreover, by the definition of  $c_{N+1}(\delta)$ , we get

$$\int_{\mathbb{R}^N} \tilde{h}\tilde{v}_{N+1} = 0 \Leftrightarrow \int_{\mathbb{R}^N} h v_{N+1} = 0 \Leftrightarrow c_{N+1}(\delta) = 0.$$

On basis of the discussion in Lemma 4.1, we can also obtain

$$\int_{\mathbb{R}^N} \tilde{h}\tilde{v}_l = \int_{\mathbb{R}^N} h v_l = 0, \quad \forall l = 1, 2, \dots, N.$$

Hence, we only need to prove that for some  $\delta_0$ ,  $c_{N+1}(\delta_0) = 0$ .

**Lemma 4.2.** *The  $\int_{\mathbb{R}^N} \tilde{h}\tilde{v}_{N+1}$  can be denoted by the following form*

$$\int_{\mathbb{R}^N} \tilde{h}\tilde{v}_{N+1} = C_N \frac{\delta}{k^{N-2s}} [\delta a_N - 1] + \frac{1}{k^{N-s}} \Theta_k(\delta).$$

Here  $\Theta_k(\delta)$  is continuous related to  $\delta$  and uniformly bounded as  $k \rightarrow \infty$ ,  $C_N = p \int_{\mathbb{R}^N} U^{p-1} v_{N+1}$ , with the positive number

$$a_N = 2^{\frac{N-2s}{2}} \lim_{k \rightarrow \infty} \frac{1}{k^{N-2s}} \sum_{j=2}^k \frac{1}{|\xi_1 - \xi_j|^{N-2s}}.$$

Obviously, by the above lemma, it is easy to see that for  $\delta$  small enough,  $\int_{\mathbb{R}^N} \tilde{h}\tilde{v}_{N+1} < 0$ , while for  $\delta$  large enough,  $\int_{\mathbb{R}^N} \tilde{h}\tilde{v}_{N+1} > 0$ . So, using the continuity property related to  $\delta$ , there exists  $\delta_0 > 0$  such that  $\int_{\mathbb{R}^N} \tilde{h}\tilde{v}_{N+1} = 0$ . The proof of this lemma is similar to Claim 1-4 in [15].

**Proof.** Note that  $\tilde{h} := \zeta_1 E + \mathcal{N}(\phi_1) = E + (\zeta_1 - 1)E + \mathcal{N}(\phi_1)$ . Then we will take three steps to discuss these terms  $E$ ,  $(\zeta_1 - 1)E$  and  $\mathcal{N}(\phi_1)$ , respectively.

**Step 1:** We will estimate the term  $\int_{\mathbb{R}^N} E\tilde{v}_{N+1}$ . Let  $\eta > 0$  be a small number, independent of  $k$ . We set

$$\int_{\mathbb{R}^N} E\tilde{v}_{N+1} = \int_{B_1} E\tilde{v}_{N+1} + \int_{\mathbb{R}^N \setminus \bigcup_{j=1}^k B_j} E\tilde{v}_{N+1} + \sum_{j \neq 1} \int_{B_j} E\tilde{v}_{N+1}. \quad (4.34)$$

where  $B_j = B(\xi_j, \frac{\eta}{k})$ ,  $\tilde{v}_{N+1}(y) := \mu^{-\frac{n-2s}{2}} v_{n+1}(\mu^{-1}(y - \xi_1))$ .

Let us consider the first term in (4.34). By changing the variables  $x = \mu y + \xi_1$ , we obtain

$$\int_{B_1} E\tilde{v}_{N+1} = \int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_1 v_{n+1}(y) dy,$$

where

$$\tilde{E}_1(y) = \mu^{\frac{n+2s}{2}} E(\xi_1 + \mu y).$$

In the region  $|y| \leq \frac{\eta}{\mu k}$ , using the expansion (2.12), we have

$$\begin{aligned} \int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_1 v_{N+1}(y) dy &= -p \sum_{j \neq 1} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(y - \mu^{-1}(\xi_j - \xi_1)) v_{N+1} \\ &\quad + p \mu^{\frac{n-2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(\xi_1 + \mu y) v_{N+1} dy \end{aligned}$$

$$\begin{aligned}
& + p \int_{B(0, \frac{\eta}{\mu k})} [(U + sV)^{p-1} - U^{p-1}] V v_{n+1} dy \\
& + \sum_{j \neq 1} \int_{B(0, \frac{\eta}{\mu k})} U^p(y - \mu^{-1}(\xi_j - \xi_1)) v_{N+1} \\
& - \mu^{\frac{N+2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} U^p(\xi_j + \mu y) v_{N+1}, \tag{4.35}
\end{aligned}$$

where

$$V = \left( - \sum_{j \neq 1} U(y - \mu^{-1}(\xi_j - \xi_1)) + \mu^{\frac{N-2s}{2}} U(\xi_1 + \mu y) \right).$$

We see that, using the Taylor expansion, for  $j \neq 1$ ,

$$\int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(y - \mu^{-1}(\xi_j - \xi_1)) v_{N+1} = 2^{\frac{N-2s}{2}} C_1 \mu^{N-2s} \frac{1}{|\hat{\xi}_j - \hat{\xi}_1|^{N-2s}} (1 + (\mu k)^2 \Theta_k(\delta)),$$

where  $C_N = \int_{\mathbb{R}^N} U^{p-1} v_{N+1}$  and  $\hat{\xi}_1 = (1, 0, \dots, 0)$  and  $\hat{\xi}_j = e^{\frac{2\pi(j-1)}{k}} \hat{\xi}_1$ . Moreover,

$$\mu^{\frac{N-2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(\xi_1 + \mu y) v_{N+1} dy = C_1 \mu^{\frac{N-2s}{2}} (1 + (\mu k)^2) \Theta_k(\delta).$$

For the third term in (4.35), using the inequality  $|(a+b)^s - a^s| \leq C|b|^s$ , we have

$$\begin{aligned}
& \int_{B(0, \frac{\eta}{\mu k})} [(U + sV)^{p-1} - U^{p-1}] V v_{n+1} dy \\
& \leq \sum_{j \neq 1} \left| \int_{B(0, \frac{\eta}{\mu k})} U^{p-1}(y - \mu^{-1}(\xi_j - \xi_1)) v_{N+1} \right| \\
& \leq C \sum_{j \neq 1} \frac{\mu^{N+2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{N-2s}} \int_{B(0, \frac{\eta}{\mu k})} \frac{1}{(1 + |y|)^{N-2s}} \\
& \leq C(\mu k)^{-2s} \sum_{j \neq 1} \frac{\mu^{N+2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{N-2s}}. \tag{4.36}
\end{aligned}$$

For the last term in (4.35), we estimate

$$\left| \mu^{\frac{N+2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} U^p(\xi_j + \mu y) v_{N+1} dy \right| \leq C \mu^{\frac{N+2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} \frac{1}{(1 + |y|)^{N-2s}} \leq C \mu^{\frac{N-2s}{2}} k^{-2s}.$$

Now for the second term in (4.34), by Hölder inequality and estimate of error term, we get

$$\begin{aligned}
\int_{EX} E \tilde{v}_{N+1} & \leq C \|(1 + |y|)^{N+2s-\frac{2N}{q}} E\|_{L^q(EX)} \cdot \|(1 + |y|)^{-N-2s-\frac{2N}{q}} \tilde{v}_{N+1}\|_{L^{\frac{q}{q-1}}(EX)} \\
& \leq C \|(1 + |y|)^{N+2s-\frac{2N}{q}} E\|_{L^q(EX)} \cdot \mu^{\frac{N-2s}{2}} \left( \int_{EX} \left[ \frac{|y - \xi_1|^{2s-N}}{(1 + |y|)^{N+2s-\frac{2N}{q}}} \right]^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \\
& \leq C k^{-2N-1+4s}.
\end{aligned}$$

Now let us consider the last term in (4.34). Set  $\tilde{E}_j = \mu^{\frac{N+2s}{2}} E(\xi_j + \mu y)$ ,  $j \neq 1$ . Performing the change of variables  $x = \mu y + \xi_j$ .

$$\begin{aligned}
\left| \int_{B_j} E \tilde{v}_{N+1} \right| &= \left| \mu^{\frac{N-2s}{2}} \int_{B(0, \eta/(\mu k))} \tilde{E}_j \tilde{v}_{N+1}(\mu y + \xi_j) dy \right| \\
&\leq C \mu^{\frac{N-2s}{2}} \|(1 + |y|)^{N+2s-\frac{2N}{q}} \tilde{E}_j\|_{L^q(B(0, \eta/(\mu k)))} \\
&\quad \times \|(1 + |y|)^{-N-2s+\frac{2N}{q}} \mu^{-\frac{N-2s}{2}} v_{N+1}(y + \mu^{-1}(\xi_j - \xi_1))\|_{L^{\frac{q}{q-1}}(B(0, \eta/(\mu k)))} \\
&\leq C \mu^{\frac{N-2s}{2}} \mu^{\frac{N-2s}{2}} (\mu k)^{-N+2s+\frac{N}{q}} \cdot \frac{\mu^{\frac{N-2s}{2}}}{|\xi_j - \xi_1|^{N-2s}} \left( \int_1^{\eta/(\mu k)} \frac{t^{N-1} dt}{t^{(N+2s-\frac{2N}{q})\frac{q}{q-1}}} \right)^{\frac{q-1}{q}} \\
&\leq C \mu^{N-2s} (\mu k)^{-N+2s+\frac{N}{q}} \frac{\mu^{\frac{N-2s}{2}}}{|\xi_j - \xi_1|^{N-2s}} (\mu k)^{2s-\frac{N}{q}}.
\end{aligned}$$

So we conclude that

$$\left| \sum_{j \neq 1} \int_{B_j} E \tilde{v}_{N+1} \right| \leq \frac{\mu^{\frac{N-2s}{2}}}{(\mu k)^{N-4s}} \left[ \mu^{N-2s} \sum_{j \neq 1} \frac{1}{|\xi_j - \xi_1|^{N-2s}} \right] \leq \frac{C k^{-2N+s}}{|\xi_j - \xi_1|^{N-2s}}.$$

**Step 2:** For  $(\zeta_1 - 1)E$ , we observe that

$$\begin{aligned}
\left| \int_{\mathbb{R}^N} (\zeta_1 - 1) E \tilde{v}_{N+1} \right| &\leq C \left| \int_{|x - \xi_1| > \eta/k} E \tilde{v}_{N+1} \right| \\
&= C \left| \int_{EX} E \tilde{v}_{N+1} \right| + C \sum_{j \neq 1} \left| \int_{|x - \xi_j| < \eta/k} E \tilde{v}_{N+1} \right|
\end{aligned}$$

In the exterior region  $EX$ , by (2.11), we observe

$$|E(y)| \leq C \frac{\mu^{\frac{N-2s}{2}}}{(1 + |y|^2)^{2s}} \sum_{j=1}^k \frac{1}{|y - \xi_j|^{N-2s}},$$

where  $C$  is a positive constant, independent of  $k$ . Moreover, in the exterior region, one has

$$\tilde{v}_{N+1} \leq C \frac{\mu^{\frac{N-2s}{2}}}{|x - \xi_1|^{N-2}}.$$

So we easily see that

$$\left| \int_{EX} E \tilde{v}_{N+1} \right| \leq C k \mu^{N-2s} \int_{\eta/k}^1 \frac{t^{N-1}}{t^{2N-4s}} dt \leq C k \mu^{N-2s} k^{N-4s}$$

and

$$\left| \int_{EX} E \tilde{v}_{N+1} \right| = \frac{1}{k^{2N-2s-1}} \Theta_k(\delta). \quad (4.37)$$

On the other hand, by changing the variables,  $\mu y = x - \xi_j$ , we have

$$\int_{|x - \xi_j| < \eta/k} E \tilde{v}_{N+1} = \mu^{\frac{N+2s}{2}} \int_{|y| \leq \eta/(k\mu)} E(\xi_j + \mu y) v_{N+1}(y + \mu^{-1}(\xi_j - \xi_1)).$$

Note that the argument of Step 2 in Lemma 2.2 implies that

$$\tilde{E}_j = \mu^{\frac{N+2s}{2}} E(\xi_j + \mu y) \leq C \frac{\mu^{\frac{N-2s}{2}}}{1 + |y|^{4s}}.$$

Furthermore, in this region, we easily obtain

$$|v_{N+1}(y + \mu^{-1}(\xi_j - \xi_1))| \leq C \frac{\mu^{N-2s} k^{N-2s}}{|j-1|^{N-2s}}.$$

Hence, we have

$$\sum_{j \neq 1} \left| \int_{|x - \xi_j| < \eta/k} E \tilde{v}_{N+1} \right| \leq k \mu^{\frac{N-2s}{2}} (k\mu)^{N-2s} \int_{|y| < \eta/(k\mu)} \frac{\mu^{\frac{N-2s}{2}}}{1 + |y|^{4s}} dy \leq C k^{-3N+2s+1}.$$

**Step 3:** By the change of variable  $x = \mu y + \xi_1$ , we have

$$\begin{aligned} \int_{\mathbb{R}^N} \mathcal{N}(\phi_1) \tilde{v}_{N+1} dx &= \int_{\mathbb{R}^N} \mathcal{N}(\phi_1) \mu^{-\frac{N-2s}{2}} v_{N+1}(\mu^{-1}(x - \xi_1)) dx \\ &= \int_{\mathbb{R}^N} \mathcal{N}(\phi_1)(\mu y + \xi_1) \mu^{\frac{N+2s}{2}} v_{N+1}(y) dy \\ &\leq C \|\mu^{\frac{N+2s}{2}} \mathcal{N}(\phi_1)(\mu y + \xi_1)\|_{**} \left( \int_{\mathbb{R}^N} \frac{1}{(1 + |y|)^{2N}} \right)^{\frac{q-1}{q}}. \end{aligned}$$

Using the estimates of  $f_1, f_2, f_3$  and  $f_4$  in Lemma 4.3, we have

$$\begin{aligned} &\|\mu^{\frac{N+2s}{2}} \mathcal{N}(\phi_1)(\mu y + \xi_1)\|_{**} \left( \int_{\mathbb{R}^N} \frac{1}{(1 + |y|)^{2N}} \right)^{\frac{q-1}{q}} \\ &\leq C k^{-\frac{3N}{2} - \frac{N}{q} - 5s} (k^{1+s - \frac{N}{2} - \frac{N}{q}} + \|\phi_1\|_*^2), \end{aligned}$$

where  $C$  is a positive constant, independent of  $k$ . Recalling that  $\mu = \delta^{\frac{2}{N-2s}} k^{-3}$  and combining the obtained estimates, we complete the proof.  $\square$

**Lemma 4.3.** For  $\tilde{h}$  given above, suppose that

$$h(y) := \mu^{\frac{N+2s}{2}} \tilde{h}(\xi_1 + \mu y)$$

satisfies  $\|h\|_{**} < \infty$ . Then (4.33) has a unique solution  $\tilde{\phi} := \tilde{T}(\tilde{h})$  which satisfies the properties (4.2) and (4.3) and

$$\int_{\mathbb{R}^N} \phi U^{p-1} v_{N+1} = 0, \text{ with } \|\phi\|_* \leq C \|h\|_{**},$$

where  $\phi(y) = \mu^{\frac{N-2s}{2}} \tilde{\phi}(\xi_1 + \mu y)$ .

**Proof.** By Lemma 4.2, we get

$$\int_{\mathbb{R}^N} h v_{N+1} = 0.$$

Now the oddness of  $v_2, v_3, \dots, v_N$  and the evenness of  $h$  imply

$$\int_{\mathbb{R}^N} h v_i = 0, \quad \forall i = 2, 3, \dots, N.$$

Next, we only need to show that  $\int_{\mathbb{R}^N} h v_1 = 0$ . Set

$$I(t) := \int_{\mathbb{R}^N} w_\mu(y - t\xi_1) \tilde{h}(y) dy,$$

where

$$w_\mu(y) = \mu^{-\frac{N-2s}{2}} U(\mu^{-1}y)$$

Then by taking the derivative of  $I(t)$ , we have

$$\begin{aligned} I'(1) &= - \int_{\mathbb{R}^N} \frac{\xi_1}{\mu} \cdot \nabla U_1(\mu^{-1}(y - t\xi_1)) h(\mu^{-1}(y - \xi_1)) \cdot \mu^{-\frac{N-2s}{2}} \cdot \mu^{-\frac{N+2s}{2}} dy \Big|_{t=1} \\ &= - \frac{\sqrt{1-\mu^2}}{\mu} \int_{\mathbb{R}^N} U_1(y) h(y) dy \\ &= - \frac{\sqrt{1-\mu^2}}{\mu} \int_{\mathbb{R}^N} h v_1. \end{aligned} \quad (4.38)$$

By making a transformation  $y \rightarrow z = \frac{y}{|y|^2}$ , we obtain

$$\begin{aligned} I(t) &:= \int_{\mathbb{R}^N} w_\mu(y - t\xi_1) |y|^{-(N+2s)} \tilde{h}(|y|^2 y) dy \\ &= \int_{\mathbb{R}^N} w_\mu(|y|^{-2} y - t\xi_1) |z|^{-(N-2s)} \tilde{h}(z) dz \\ &= \int_{\mathbb{R}^N} \left( \frac{\mu}{\mu^2 + t^2 |\xi_1|^2} \right)^{\frac{N-2s}{2}} c_{n,s}^{\frac{N-2s}{4s}} \cdot \left[ \left| y - \frac{t\xi_1}{\mu^2 + t^2 |\xi_1|^2} \right|^2 + \frac{\mu^2}{(\mu^2 + t^2 |\xi_1|^2)^2} \right]^{-\frac{N-2s}{2}} \tilde{h}(y) dy \\ &= \int_{\mathbb{R}^N} w_{\mu(t)}(y - s(t)\xi_1) \tilde{h}(y) dy, \end{aligned} \quad (4.39)$$

where

$$\mu(t) = \frac{\mu}{\mu^2 + t^2 |\xi_1|^2}, \quad s(t) = \frac{t}{\mu^2 + t^2 |\xi_1|^2}.$$

Through taking the derivative on both side of (4.39), one has

$$\begin{aligned} I'(1) &= \left[ \int_{\mathbb{R}^N} \mu'(t) \frac{\partial(w_\mu(y - s(t)\xi_1))}{\partial \mu} \tilde{h}(y) dy - \xi_1^1 \int_{\mathbb{R}^N} s'(t) \frac{\partial(w_{\mu(t)}(y - s(t)\xi_1))}{\partial y_1} \tilde{h}(y) dy \right] \Big|_{t=1} \\ &= 2\mu^2 \int_{\mathbb{R}^N} \tilde{v}_{N+1}(y) \tilde{h}(y) dy - \frac{(2\mu^2 - 1)\sqrt{1-\mu^2}}{\mu} \int_{\mathbb{R}^N} h v_1 \\ &= \frac{(2\mu^2 - 1)\sqrt{1-\mu^2}}{\mu} \int_{\mathbb{R}^N} h v_1. \end{aligned} \quad (4.40)$$

So (4.38) and (4.40) imply that

$$I'(1) = - \frac{\sqrt{1-\mu^2}}{\mu} \int_{\mathbb{R}^N} h v_1 = \frac{(2\mu^2 - 1)\sqrt{1-\mu^2}}{\mu} \int_{\mathbb{R}^N} h v_1.$$

Obviously, this shows that

$$\int_{\mathbb{R}^N} h v_1 = 0.$$

Now Lemma 3.2 implies that Eq. (4.33) has a unique solution  $\tilde{\phi} := \tilde{T}(\tilde{h})$  which is even with respect to each of the variables  $y_2, y_3, \dots, y_N$  satisfying

$$\|\tilde{\phi}\|_* = \|\tilde{T}(\tilde{h})\|_* \leq C\|\tilde{h}\|_{**}.$$

But  $\tilde{h}$  in Eq. (4.33) does not satisfy the property (4.15), so the discussion of Lemma 4.1 can not be applied to obtain the existence directly. We must prove that the operator  $\tilde{T}$  is a contraction mapping again.

In the following, for convenience sake, the term  $\zeta_1 E + \mathcal{N}(\phi_1)$  is split into five terms. And we estimate these terms one by one. Set

$$\begin{aligned} f_1 &:= p\zeta_1(U_1^{p-1} - |U_*|^{p-1}) \cdot \tilde{\phi}_1, & f_2 &:= p(1 - \zeta_1)U_1^{p-1}\tilde{\phi}_1, \\ f_3 &:= -p\zeta_1 U_*^{p-1}\psi(\phi_1), & f_4 &:= \xi_1 N\left(\sum_{j=1}^k \tilde{\phi}_j + \psi(\phi_1)\right), & f_5 &:= \zeta_1 E, \end{aligned}$$

and

$$\tilde{f}_i(y) = \mu^{\frac{N+2s}{2}} f_i(\xi_1 + \mu y), \quad i = 1, \dots, 5.$$

Then

$$\tilde{h} = \sum_{i=1}^5 \tilde{f}_i.$$

Due to the cut-off function  $\zeta_1$ , we see that

$$\text{supp } f_j \subset \{y : |y - \xi_1| < \eta/k\} =: IN_1 \subset IN, \quad j = 1, 3, 4, 5.$$

For  $f_1$ , we get

$$\begin{aligned} |\tilde{f}_1| &\leq \left| p \left| U(y) + \sum_{j=2}^k U(y + \mu^{-1}(\xi_1 - \xi_j)) - \mu^{\frac{N-2s}{2}} U(\xi_1 + \mu y) \right|^{p-1} - pU^{p-1}(y) \right| \cdot |\phi_1(y)| \\ &\leq C \left| \sum_{j=2}^k U(y + \mu^{-1}(\xi_1 - \xi_j)) + \mu^{\frac{N-2s}{2}} U(\xi_1 + \mu y) + U(y) \right|^{p-2} \\ &\quad \cdot |\mu^{\frac{N-2s}{2}} U(\xi_1 + \mu y) + U(y)| \cdot U(y) \cdot \|\phi_1\|_* \\ &\leq CU^{p-1}(y) \mu^{\frac{N-2s}{2}} \|\phi_1\|_* \leq C \frac{\mu^{\frac{N-2s}{2}}}{1 + |y|^{4s}} \|\phi_1\|_*. \end{aligned}$$

Hence we take the the same argument of Step 2 in Lemma 2.2 and infer

$$\|f_1\|_{**} = \|f_1\|_{**(IN_1)} \leq C\|\phi_1\|_* k^{1+s-\frac{N}{2}-\frac{N}{q}}. \quad (4.41)$$

For  $f_2$ , we see that

$$|\tilde{f}_2(y)| = |\zeta_1(\mu y + \xi_1) - 1| \cdot U^{p-1} \cdot |\phi_1| \leq C|U|^p \|\phi_1\|_*.$$

Thus,

$$\|f_2\|_{**} \leq C \left[ \int_{|y-\xi_1|>\eta/k} (1 + |y|)^{(N+2s)q-2N} \mu^{-\frac{N+2s}{2}q} \left| \tilde{f}_2^q \left( \frac{y - \xi_1}{\mu} \right) \right| dy \right]^{1/q}$$

$$\begin{aligned}
&\leq C \left[ \mu^{\frac{q(N+2s)}{2}} \cdot \left( \int_{\eta/k}^1 r^{N-1-(N-2s)pq} dr + \int_1^\infty r^{(N+2s)q-2N-(N-2s)pq+N-1} dr \right) \right]^{1/q} \|\phi_1\|_* \\
&\leq C \mu^{\frac{N+2s}{2}} k^{(N+2s)-\frac{N}{q}} \|\phi_1\|_* + C \mu^{\frac{q(N+2s)}{2}} \|\phi_1\|_* < C k^{-\frac{3(N+2s)}{2}} \|\phi_1\|_*.
\end{aligned} \tag{4.42}$$

Analogously, applying the estimate of  $\psi$  in Lemma 4.1, we have

$$\begin{aligned}
|\tilde{f}_3| &\leq C U^{p-1} \mu^{\frac{N-2s}{2}} \|\psi(\phi_1)\|_\infty \\
&\leq C U^{p-1} \mu^{\frac{N-2s}{2}} \|\psi(\phi_1)\|_* \\
&\leq C \mu^{\frac{N-2s}{2}} \frac{1}{1+|y|^{4s}} (k^{1+s-\frac{N}{2}-\frac{N}{q}} + \|\phi_1\|_*^2).
\end{aligned}$$

and

$$\|f_3\|_{**} \leq C k^{-\frac{N}{q}-2s} (k^{1+s-\frac{N}{2}-\frac{N}{q}} + \|\phi_1\|_*^2). \tag{4.43}$$

Now, for  $f_4$ , noting that

$$\tilde{N} = |V_* + \hat{\phi}|^{p-1} (V_* + \hat{\phi}) - |V_*|^{p-1} V_* - p|V_*|^{p-1} \hat{\phi}.$$

where  $\hat{\phi}(y) := \mu^{\frac{N-2s}{2}} \phi(\xi_1 + \mu y)$ , and

$$V_*(y) = U(y) + \sum_{j=2}^k U(y + \mu^{-1}(\xi_1 - \xi_j)) - \mu^{\frac{N-2s}{2}} U(\xi_1 + \mu y).$$

So, for  $\phi = \sum_{j=1}^k \tilde{\phi}_j + \psi(\phi_1)$ , we have

$$|\tilde{f}_4(y)| \leq C U^{p-1} \mu^{\frac{N-2s}{2}} \left[ \|\phi_1\|_* + (\|\phi_1\|_*^2 + k^{1+s-\frac{N}{2}-\frac{N}{q}}) \right],$$

thus

$$\|f_4\|_{**} \leq C k^{-\frac{N}{q}-2s} \left[ \|\phi_1\|_* + (\|\phi_1\|_*^2 + k^{1+s-\frac{N}{2}-\frac{N}{q}}) \right]. \tag{4.44}$$

It follows from the estimate of the error term  $E$  that

$$\|f_5\|_{**} \leq C k^{1+s-\frac{N}{2}-\frac{N}{q}}. \tag{4.45}$$

Combining the obtained estimates for  $f_1, \dots, f_5$ , for all  $\hat{\phi}, \hat{\phi}_1, \hat{\phi}_2 \in B_\rho(0) \subset X$ , we have

$$\|\mathcal{M}(\hat{\phi})\|_* \leq C \sum_{i=1}^5 \|f_{i=1}(\hat{\phi})\|_{**} \leq C k^{-\frac{N}{q}-2s} (\|\phi_1\|_* + \|\hat{\phi}\|_*^2), \tag{4.46}$$

and

$$\begin{aligned}
\|\mathcal{M}(\hat{\phi}_1) - \mathcal{M}(\hat{\phi}_2)\|_* &\leq C \sum_{i=1}^5 \|f_i(\hat{\phi}_1) - f_i(\hat{\phi}_2)\|_{**} \\
&\leq C k^{-\frac{N}{q}-2s} (\|\hat{\phi}_1\|_* + \|\hat{\phi}_2\|_*) \|\hat{\phi}_1 - \hat{\phi}_2\|_* \\
&=: \lambda \|\hat{\phi}_1 - \hat{\phi}_2\|_*, \quad \text{with } \lambda < 1.
\end{aligned}$$

Hence  $\mathcal{M}$  is a contraction mapping from  $B_\rho(0)$  to  $B_\rho(0)$ , for  $k$  large enough. By the Banach fixed point theorem, there exists a unique solution  $\hat{\phi}_1$  of Eq. (4.33).  $\square$



## 5 Proof of main result

We assume that the original problem (1.1) admits a solution of the form:

$$U = U_*(y) + \phi(y).$$

Then problem (1.1) is converted into Eq. (2.5). Through introducing the cut-off functions, and assuming that  $\phi = \sum_{j=1}^k \tilde{\phi}_j + \psi$ , Eq. (2.5) is turned into a system of equations of  $\tilde{\phi}_j, j = 1, 2, \dots, k$ , and  $\psi$  (see (4.5)). Hence, we only need to prove the existences of  $\tilde{\phi}_j, j = 1, 2, \dots, k$ , and  $\psi$ , which are done in Section 4 by Banach fixed point theorem. So for  $k > k_0$ , the sign-changing solutions  $u_k = U_*(y) + \sum_{j=1}^k \tilde{\phi}_j + \psi$  for problem (1.1) are obtained.

In short, the outline of our proofs is as follows:

$$\begin{aligned} \text{Eq. (1.1)} &\xleftrightarrow{u=U_*+\phi} \text{Eq. (2.5)} \xleftrightarrow{\phi=\sum_{j=1}^k \tilde{\phi}_j+\psi} \text{Eq. (4.5)} \iff \\ &\begin{cases} \text{Eq. (4.6)} & \text{unique existence} \Leftarrow \text{Banach fixed point theorem;} \\ \text{Eq. (4.33)} & \text{unique existence} \Leftarrow \text{Banach fixed point theorem.} \end{cases} \end{aligned}$$

The proof is completed.

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